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Letter to the Editor

Homogenization of a waffle membrane

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1. Introduction

Many papers have been devoted to the analysis of ribbed plates and shells [1–5]. The method of homogenization is applied [3–5], as a rule, to constructions with unidirectional reinforcing elements. In contrast, the waffle membrane and some problems not treated in the mentioned references are considered, for instance, the calculation of boundary layers and membranes with concentrated loads, are investigated.

A square membrane ($|x| \leqslant L, |y| < L$), reinforced in two main directions by the fibres is considered. The following equilibrium equation holds between fibres:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = Q(x, y). \tag{1}$$

Here u is displacement and $Q(x, y)$ is loading.

The distance between neighbouring fibres is equal to l. It is assumed that there is no interaction between the fibres. Note that a generalization of the approach into a rectangular membrane with unequal distances between fibres in its two directions, as well as a different stiffness of the fibres can be easily realized.

Assuming $L \gg l$ one may introduce a small parameter $\varepsilon = l/L$.

First, it is assumed that the loading $Q(x, y)$ changes slowly. This assumption can be clarified in the following way. Let $Q(x, y)$ be a truncated double Fourier series. If the minimal period of the Fourier series harmonics is essentially larger that the distance between fibres, then one has to deal with the so-called slow load.

Conjugation conditions (conditions of bonds between membrane and fibres) can be written in the following form:

$$u^+ = u^-, \quad \left(\frac{\partial u}{\partial X}\right)^+_X - \left(\frac{\partial u}{\partial X}\right)^-_X = D\frac{\partial^2 u}{\partial Y^2}\bigg|_{X=kl},$$

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$$u^+ = u^-, \quad \left(\frac{\partial u}{\partial Y}\right)^+_Y - \left(\frac{\partial u}{\partial Y}\right)^-_Y = D\left.\frac{\partial^2 u}{\partial X^2}\right|_{Y=kl}, \quad k = 0, \pm 1, \ldots, n. \tag{2}$$

In the above the following notation is used:

$$(\ldots)^{\pm}_X = \lim_{X \to kl \pm 0} (\ldots),$$

$$(\ldots)^{\pm}_Y = \lim_{Y \to kl \pm 0} (\ldots), \quad k = 0, \pm 1, \ldots, n, \tag{3}$$

D is relative rigidities of fibres.

The following boundary conditions are applied:

$$u = 0 \quad \text{for } X = \pm L, Y = \pm L. \tag{4}$$

Let us introduce "slow" (x, y) and "fast" (ξ, η) variables via the relations:

$$x = X/L, \quad y = Y/L, \quad \xi = X/l, \quad \eta = Y/l.$$

The following new relations hold for the derivatives:

$$\frac{\partial}{\partial X} = \frac{1}{L}\left(\frac{\partial}{\partial x} + \varepsilon^{-1}\frac{\partial}{\partial \xi}\right); \quad \frac{\partial}{\partial Y} = \frac{1}{L}\left(\frac{\partial}{\partial y} + \varepsilon^{-1}\frac{\partial}{\partial \eta}\right). \tag{5}$$

The sought function is approximated by the series:

$$u = u_0(x, y) + \varepsilon^2[u_1(x, y, \xi, \eta)] + \varepsilon^2[u_2(x, y, \xi, \eta)] + \cdots . \tag{6}$$

Observe that the function u_i is periodic, and hence

$$u_i(x, y, \xi + 1, \eta) = u_i(x, y, \xi, \eta),$$

$$u_i(x, y, \xi, \eta + 1) = u_i(x, y, \xi, \eta), \quad i = 1, 2, \ldots . \tag{7}$$

Substituting Eq. (6) into Eqs. (1)–(4) and taking into account (5), one obtains

$$\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 u_1}{\partial \xi^2} + \frac{\partial^2 u_1}{\partial \eta^2} = q(x, y), \tag{8}$$

$$2\frac{\partial^2 u_1}{\partial \xi \partial x} + 2\frac{\partial^2 u_1}{\partial \eta \partial y} + \frac{\partial^2 u_2}{\partial \xi^2} + \frac{\partial^2 u_2}{\partial \eta^2} = 0, \tag{9}$$

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + 2\frac{\partial^2 u_2}{\partial \xi \partial x} + +2\frac{\partial^2 u_2}{\partial \eta \partial y} + \frac{\partial^2 u_3}{\partial \xi^2} + \frac{\partial^2 u_3}{\partial \eta^2} = 0, \tag{10}$$

$$\vdots$$

$$\left(\frac{\partial u_1}{\partial \xi}\right)^+_\xi - \left(\frac{\partial u_1}{\partial \xi}\right)^-_\xi = d\left(\frac{\partial^2 u_0}{\partial y^2} + \left.\frac{\partial^2 u_1}{\partial \eta^2}\right|_{\xi=k}\right), \tag{11}$$

$$\left(\frac{\partial u_1}{\partial \eta}\right)^+_\eta - \left(\frac{\partial u_1}{\partial \eta}\right)^-_\eta = d\left(\frac{\partial^2 u_0}{\partial x^2} + \left.\frac{\partial^2 u_1}{\partial \xi^2}\right|_{\eta=k}\right), \tag{12}$$

$$\left(\frac{\partial u_2}{\partial \xi}\right)_\xi^+ - \left(\frac{\partial u_2}{\partial \xi}\right)_\xi^- = d\left(2\frac{\partial^2 u_1}{\partial \eta \partial y} + \frac{\partial^2 u_2}{\partial \eta^2}\right)\bigg|_{\xi=k}, \tag{13}$$

$$\left(\frac{\partial u_2}{\partial \eta}\right)_\eta^+ - \left(\frac{\partial u_2}{\partial \eta}\right)_\eta^- = d\left(2\frac{\partial^2 u_1}{\partial \xi \partial x} + \frac{\partial^2 u_2}{\partial \xi^2}\right)\bigg|_{\eta=k}, \tag{14}$$

$$\left(\frac{\partial u_3}{\partial \xi}\right)_\xi^+ - \left(\frac{\partial u_3}{\partial \xi}\right)_\xi^- = d\left(\frac{\partial^2 u_1}{\partial y^2} + 2\frac{\partial^2 u_2}{\partial \eta \partial y} + \frac{\partial^2 u_3}{\partial \eta^2}\right)\bigg|_{\xi=k}, \tag{15}$$

$$\left(\frac{\partial u_3}{\partial \eta}\right)_\eta^+ - \left(\frac{\partial u_3}{\partial \eta}\right)_\eta^- = d\left(\frac{\partial^2 u_1}{\partial x^2} + 2\frac{\partial^2 u_2}{\partial \xi \partial x} + \frac{\partial^2 u_3}{\partial \xi^2}\right)\bigg|_{\eta=k}, \quad k = 0, \pm 1, \pm 2, \ldots, \pm n, \tag{16}$$

$$\vdots$$

$$u_0 = 0 \quad \text{for } x = \pm L,\ y = \pm L, \tag{17}$$

$$u_i = 0 \quad \text{for } x = \pm L,\ \xi = \pm \varepsilon^{-1} L,\ y = \pm L,\ \eta = \pm \varepsilon^{-1} L, \tag{18}$$

where $q(x, y) = Q(x, y)L^2$; $d = D/L$; $(\ldots)_\xi^\pm = \lim_{\xi \to k \pm 0}(\ldots)$; $(\ldots)_\eta^\pm = \lim_{\eta \to k \pm 0}$; $k = 0, \pm 1, \pm 2, \ldots n$.

In the above $(2n + 1)l = 2L$. A construction of the asymptotes essentially depends on the order of the parameter d. For estimation of the order of the parameter d, the following new parameter α is introduced via the relation $d \sim \varepsilon^\alpha$. The analysis carried out yielded three different cases:

(i) $\alpha > 1$. In this case the membrane can be treated as a body without fibres. Their influence appears in the successive approximation of a perturbation approach.
(ii) $\alpha < 1$. In this case, one deals with the membrane clamped along the fibres in the first approximation.
(iii) $\alpha = 1$. This case is the most interesting one. In order to analyse it properly the following periodical cell $(0 \leqslant \xi \leqslant 1,\ 0 \leqslant \eta \leqslant 1)$ is introduced.

Conjugation conditions (11)–(16), taking into account the periodicity conditions (7), are sought in the form

$$\frac{\partial u_1}{\partial \xi}\bigg|_{\xi=1} - \frac{\partial u_1}{\partial \xi}\bigg|_{\xi=0} = d\left(\frac{\partial^2 u_0}{\partial y^2} + \underline{\frac{\partial^2 u_1}{\partial \eta^2}\bigg|_{\xi=0}}\right), \tag{19}$$

$$\frac{\partial u_1}{\partial \eta}\bigg|_{\eta=1} - \frac{\partial u_1}{\partial \eta}\bigg|_{\eta=0} = d\left(\frac{\partial^2 u_0}{\partial x^2} + \underline{\frac{\partial^2 u_1}{\partial \xi^2}\bigg|_{\eta=0}}\right). \tag{20}$$

Now, integrating Eq. (8) with respect to ξ and η in the intervals of the cell diameters, i.e., from 0 to 1, and accounting for relations (19) and (20), the following differential equation is obtained:

$$(1+d)\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2}\right)=q(x,y). \tag{21}$$

The underlined terms in relations (19) and (20) are governed by the concentrated functions distributed along lines and they vanish during the integration process.

From a physical point of view, Eq. (21) is obtained as a result of a uniform distribution of the fibres stiffness along the membrane.

It is worth noting that the "fast part" of the solution is obtained first. After that the "slow part" has been determined, in contrast with many other homogenization problems.

A brief comment related to the construction of the fast solution components is now given. Since, in general, the solution obtained does not satisfy the boundary conditions, it is necessary to improve it via a boundary layer approach. In the latter case one may apply the ideas related to a three-phase modelling migrated from the theory of composites [6].

In the beginning the right-hand side zone is separated (Fig. 1). This zone yields a periodically repeated cell, and the remaining membrane part is substituted by an averaged system.

A similar idea can be also applied during computation with inclusion of a point force (Fig. 2). In the latter case, a cell with a point force is separated, and the influence of the remaining part of the membrane is estimated via the homogenized relations.

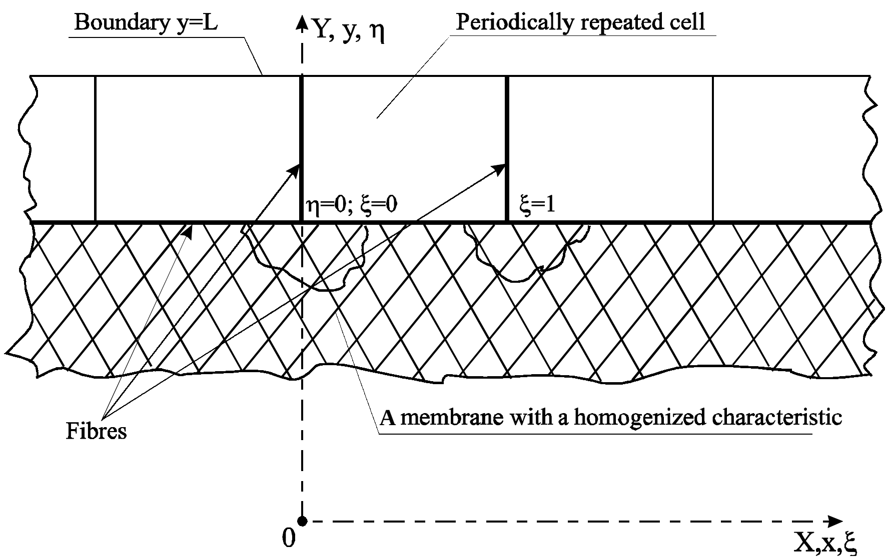

Fig. 1. Diagram of a boundary layer zone.

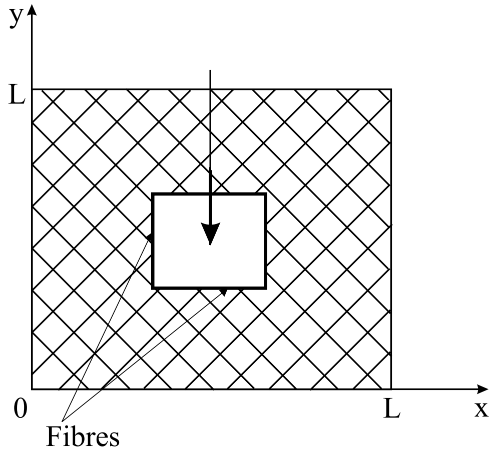

Fig. 2. A point-force action.

2. Conclusions

The results obtained give a good prognosis for a generalization of this approach into a dynamical case.

Besides, a generalization of the presented technique into the analysis of waffle-type plate and shells is expected to yield only technical problems, which can be omitted.

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References

[1] I.Ya. Amiro, V.A. Zarutsky, Studies of the dynamics of ribbed shells, Soviet Applied Mechanics 17 (11) (1981) 949–962.
[2] I.Ya. Amiro, V.A. Zarutsky, Taking the discrete rib spacing into account when studying the stress–strained state, vibration and stability of ribbed shells (Review), International Journal of Applied Mechanics 34 (4) (1998) 299–314.
[3] J. Awrejcewicz, I.V. Andrianov, L.I. Manevitch, Asymptotic Approaches in Nonlinear Dynamics: New Trends and Applications, Springer, Heidelberg, 1998.
[4] I.V. Andrianov, J. Awrejcewicz, Homogenization procedure in the theory of periodically non-homogeneous plates and shells, in: M. Papadrahis, A. Samartin, E. Onate (Eds.), IASS-IACM 2000, Fourth International Colloquium on Computation of Shell & Spatial Structures, Chania-Crete, Greece, 4–7 June 2000, pp. 1–18.
[5] I.V. Andrianov, L.I. Manevitch, V.O. Oshmyan, Mechanics of Periodically Heterogeneous Structures, Springer, Berlin, 2002.
[6] R.M. Christensen, Mechanics of Composite Materials, Wiley, New York, 1979.